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Analytical and Numerical Study of Temperature Fields in Cylindrical Thermoelements with Internal Heat Sources

Using the method of finite integral transformations, an analytical solution was provided to the problem of non-stationary thermal conductivity of a bounded cylinder with continuously operating heat sources placed in a medium with a constant temperature. An approach for numerical study of thermal fields in cylindrical structures with internal heat sources, with regard to the influence of external conditions, is proposed. Numerical implementation of the analytical solution, numerical simulation of heat distribution, and 3D visualization of the thermodynamic behaviour of the system are carried out. The numerical simulation technique is used to predict the temperature mode in thermoelectric structures with internal heat sources, which is the basis for the engineering design of efficient thermoelectric systems.

Keywords: non-stationary thermal conductivity, finite integral transformation method, boundary conditions, thermoelectric generator, thermoelectric effects, internal heat sources, temperature field, numerical simulation.

Introduction

The development of rational methods and algorithms for numerical solution of non-stationary problems of thermophysics with temperature dependence of thermophysical properties is extremely important and relevant for the design of modern industrial facilities.

The work develops an algorithm that allows describing the temperature distribution in the cylindrical region of the studied structure under the action of internal heat sources and the influence of external conditions. The numerical simulation technique can be effectively used to predict the temperature mode in thermoelectric generators (TEGs), which have the following features:

- internal heat sources: radioisotope, electrical or chemical;

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- compact dimensions (often cylindrical or close to such);
- sensitivity to overheating or local heat accumulation.

The numerical simulation conducted in the work can be effectively applied to predict thermal processes in TEGs, where heat sources are placed inside a cylindrical region; to optimize the geometry and materials of TEGs to ensure the highest thermoelectric efficiency at an allowable temperature; to assess the durability and stability of materials, taking into account heat accumulation in the centre and potential overheating.

In particular, the proposed approach for analytical and numerical study of thermal fields in cylindrical structures with internal heat sources is suitable for calculating temperature processes in real engineering systems, such as thermoelectric generators (TEGs) [1-5], infrared sensors, etc., and studying processes that generate heat fluxes inside the structure and lead to thermoelectric effects.

Thus, the resulting solution can be a promising basis for the engineering design of efficient thermoelectric systems.

We will connect the research carried out in the work with the operation of a thermoelectric generator (TEG) (Fig. 1), establish the nature of the processes that generate temperature gradients inside the structure, which leads to the occurrence of thermoelectric effects.

Thermoelectric generators (TEGs), which are based on internal heat sources, use temperature gradients that arise directly inside the structure of a material or device. This allows for efficient conversion of thermal energy into electrical energy due to thermoelectric effects, in particular the Seebeck effect. Below are the main types of such thermoelectric generators, their operating principle, examples and the role of internal heat sources.

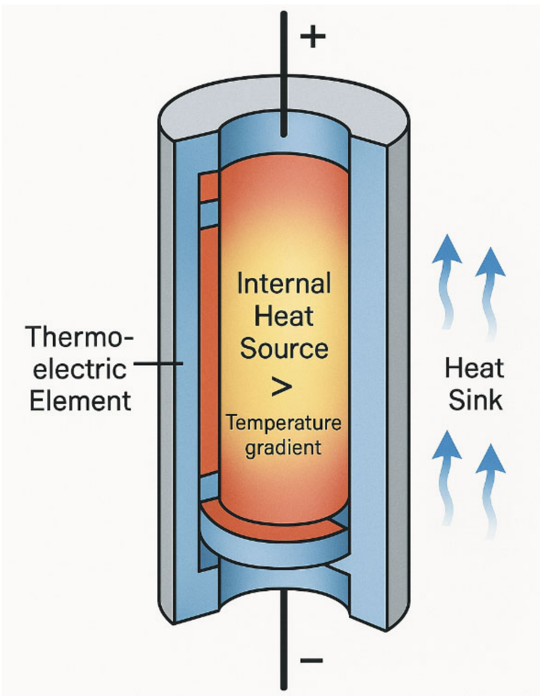


Fig. 1. Schematic of a thermoelectric generator

Thermoelectric generators with built-in radioisotope sources (RTG)

Inside the generator is a radioactive substance (e.g., plutonium-238) that releases heat as it decays. This heat creates a temperature gradient between the heated interior and the cooled exterior. This gradient creates a thermo-emf that generates a current between the hot and cold ends of the thermocouple [6,7].

Electrically heated thermoelements (electrically activated TEGs)

The source is an electric current supplied from the outside, but the heating itself occurs inside the structure.

Chemical thermoelectric generators

Heat is released as a result of an exothermic chemical reaction (e.g., oxidation of a metal or organic substance). The generator is designed so that this reaction occurs inside the housing, where the thermoelements are located.

Thermoelectric generators with internal photon or laser sources

Inside the thermoelement, a source of infrared or laser radiation is placed, which is directed at an absorbing layer. This layer converts the light energy into heat, forming a temperature gradient.

The role of the internal heat source in TEG:

– creates a stable temperature field inside a confined structure; provides a constant supply of heat necessary for continuous generation of electric current;

– forms a heat flux through the thermoelectric element, which ensures the operation of the generator.

In all cases, internal heat sources are a key factor in ensuring a stable temperature gradient, without which the occurrence of thermoelectric current is impossible.

Analytical solution to the problem of non-stationary thermal conductivity

Using the method of finite integral transformations [8], we will solve the problem of non-stationary thermal conductivity of a bounded cylinder with continuously operating heat sources therein.

Problem statement: Consider a homogeneous solid bounded cylinder of circular cross-section with radius R and height l with active internal heat sources placed in a medium with a time-varying temperature $\widehat{T}_C(t)$. At the boundaries of the cylinder, free heat exchange occurs according to Newton's law. Inside the cylinder, time-dependent heat sources with a volumetric power density q_V operate. Let us introduce a temperature reference from $\widehat{T}_C(t)$, i.e. $T=\widehat{T}-\widehat{T}_C$, then in the coordinate system (r,z) the mathematical formulation of the problem can be written as:

$$\frac{1}{a}\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}+\frac{q_V(t)}{\lambda}-\frac{1}{a}\frac{\partial \widehat{T}_C(t)}{\partial t}, \tag{1}$$

$$(0<r<R,\ 0<z<l,\ 0<t<\infty)$$

the initial condition specifies the initial temperature T^O inside the cylinder

$$T(r,z,0)=T^O,\quad 0<r<R,\quad 0<z<l; \tag{2}$$

The boundary conditions reflect the heat exchange with the environment with temperature T_C and specify the heat flux through the base of the cylinder at $z=0$:

$$\frac{\partial T(0,z,t)}{\partial r}=0,\quad 0<z<l,\quad 0<t<\infty; \tag{3}$$

$$\frac{\partial T(R,z,t)}{\partial r}+h_1T(R,z,t)=0,\quad 0<z<l,\quad 0<t<\infty; \tag{4}$$

$$\frac{\partial T(r,0,t)}{\partial z}-h_2T(r,0,t)=0,\quad 0<r<R,\quad 0<t<\infty; \tag{5}$$

$$\frac{\partial T(r,l,t)}{\partial z}+h_3T(r,l,t)=0,\quad 0<r<R,\quad 0<t<\infty. \tag{6}$$

Here r, z [m] – transverse and longitudinal coordinates, respectively; t [s] – time; T, T^O [K] – current and initial temperatures; a [m^2·s^{-1}] – thermal diffusivity coefficient; q_V [W/m^3] – volumetric power density of internal heat sources; λ [W/(m·K)] – thermal conductivity coefficient; $h_i=\frac{\alpha_i}{\lambda}$ [m^{-1}], α_i [W/(m^2·K)] – heat transfer coefficient, i = 1, 2, 3.

If we introduce the following scales: T_M – for temperature, R – for linear dimensions, R^2/a – for time, q_{V0} – for volumetric heat release density, then problem (1) – (6) will take on the form

$$\frac{\partial u}{\partial \tau}=\frac{\partial^2 u}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial u}{\partial \rho}+\frac{\partial^2 u}{\partial Z^2}+Q(\tau)-\frac{\partial \hat{u}_C(\tau)}{\partial \tau}, \tag{7}$$

$$(0<\rho<1,\ 0<Z<\Lambda,\ 0<\tau<\infty)$$

$$u(\rho,Z,0)=u^O,\ 0<\rho<1,\ 0<Z<\Lambda; \tag{8}$$

$$\frac{\partial u(0,Z,\tau)}{\partial \rho}=0,\ 0<Z<\Lambda,\ 0<\tau<\infty; \tag{9}$$

$$\frac{\partial u(1,Z,\tau)}{\partial \rho}+Bi_1u(1,Z,\tau)=0,\ 0<Z<\Lambda,\ 0<\tau<\infty; \tag{10}$$

$$\frac{\partial u(\rho,0,\tau)}{\partial Z}-Bi_2u(\rho,0,\tau)=0,\ 0<\rho<1,\ 0<\tau<\infty; \tag{11}$$

$$\frac{\partial u(\rho,\Lambda,\tau)}{\partial Z}+Bi_3u(\rho,\Lambda,\tau)=0,\ 0<\rho<1,\ 0<\tau<\infty. \tag{12}$$

Here, $\rho = r/R$; $Z = z/R$; $\Lambda = \lambda/R$; $u = T/T_M$; $u^O = T^O/T_M$; $\hat{u}_C = \hat{T}_C/T_M$; $T_M = q_{V0}R^2/\lambda$; $\tau = F_O = at/R^2$ (F_O – Fourier number); $h_i R = \alpha_i R/\lambda = Bi_i$, $i = 1,2,3$ (Bi – Biot number); $Q(\tau) = q_V(t)/q_{VO}$.

Let us solve problem (7)–(12) using the method of finite integral transformations.

Following certain requirements, to eliminate differential expressions for Z we obtain the core of the direct transformation in the form

$$\bar{K}_{\mu_{nn}}(Z) = \frac{1}{C_{\mu_n}}(\cos\mu_n Z + \frac{Bi_2}{\mu_n}\sin\mu_n Z). \tag{13}$$

The normalizing divisor is equal to

$$C_{\mu_n} = \frac{\Lambda}{2}\left(1 + \frac{Bi_2^2}{\mu_n^2}\right) + \frac{(Bi_2 + Bi_3)(\mu_n^2 + Bi_2 Bi_3)}{2\mu_n^2(\mu_n^2 + Bi_3^2)},$$

and the eigenvalues μ_n (n = 1, 2, 3) are the positive roots of the characteristic equation

$$\mathrm{ctg}\mu\Lambda = \frac{\mu^2 - Bi_2 Bi_3}{\mu(Bi_2 + Bi_3)}. \tag{14}$$

Using the integral transformation in the interval $0 < Z < \Lambda$ with the core (13), we reduce problem (7)–(12) to the form

$$\frac{\partial\bar{u}}{\partial\tau} = \frac{\partial^2\bar{u}}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\bar{u}}{\partial\rho} - \mu_n^2\bar{u} + \bar{f}(\tau), \tag{15}$$

here the dash indicates the integral images of the initial quantities u, $u(0)$ and $f(\tau) = Q(\tau) - \frac{\partial\hat{u}_C}{\partial\tau}$, i.e.

$$\bar{u} = \int_0^\Lambda \bar{K}_{\mu_n} u dZ;\ \bar{u}(0) = A_n u^O;\ \bar{f}(\tau) = A_n f(\tau), \tag{16}$$

where $A_n = \frac{1}{C_{\mu_n}\mu_n^2}\left((-1)^{n+1} Bi_3\sqrt{\frac{\mu_n^2 + Bi_2^2}{\mu_n^2 + Bi_3^2}}\right) + Bi_2$.

We exclude differential operations on ρ. The core of the direct transformation in this case has the form

$$\tilde{K}_{\nu_k}(\rho) = \frac{\rho}{C_{\nu_k}} J_0(\nu_k\rho), \tag{17}$$

normalizing divisor $C_{\nu_k} = \int_0^1 J_o^2(\nu_k\rho)\rho d\rho = \frac{1}{2}\left(J_0^2(\nu_k) + J_1^2(\nu_k)\right)$;

$J_0(x)$, $J_1(x)$ – Bessel functions of the first kind of zero and first order; the eigenvalues ν_k^2 are the positive roots of the characteristic equation

$$\frac{J_0(\nu)}{J_1(\nu)} = \frac{\nu}{Bi_1}. \tag{18}$$

By directly transforming problem (15), (16) with core (17) in the interval $0 < \rho < 1$, we obtain the ordinary inhomogeneous first-order differential equation

$$\frac{d\tilde{u}}{d\tau} + (\mu_n^2 + \nu_k^2)\tilde{u} = \tilde{f}(\tau),$$

here $\tilde{u} = \frac{1}{C_{\nu_k}} \int_0^1 \overline{u}(\rho, \tau) J_0(\nu_k \rho) \rho d\rho$, $\tilde{u}(0) = A_n A_k u^O$, $\tilde{f}(\tau) = A_n A_k f(\tau)$,

$$A_k = \frac{2Bi_1}{J_0(\nu_k)(\nu_k^2 + Bi_1^2)}, \tag{19}$$

whose solution, obtained by the method of variation of the constant, has the form

$$\tilde{u}(\tau) = e^{-(\mu_n^2+\nu_k^2)} \left(\int_0^\tau \tilde{f}(\tau') e^{(\mu_n^2+\nu_k^2)\tau'} d\tau' + \tilde{u}(0) \right).$$

It should be noted that in the obtained result, the integral images of the double transformation of the original quantities are marked with a tilde.
By performing the inverse transformation, we find the final solution to the problem

$$u(\rho, Z, \tau) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} A_n A_k e^{-(\mu_n^2+\nu_k^2)\tau} (\cos \mu_n Z + \frac{Bi_2}{\mu_n} \sin \mu_n Z) \times$$
$$\times J_0(\nu_k \rho) \left(\int_0^\tau f(\tau') e^{(\mu_n^2+\nu_k^2)\tau'} d\tau' + u^0 \right) \tag{20}$$

The resulting solution is an alternating sign series. Since for alternating sign series the sum of the rejected terms in absolute value does not exceed the first of these terms, then with a relative error $\varepsilon = 0.001$ for small time intervals ($\tau \sim 10^{-4}$) their number is $n = k = 30$. With increasing time, the number of terms decreases. Thus, with $\tau > 0.5$ the specified accuracy is achieved by retaining only four terms of the series.

Therefore, for the analytical solution of the problem, the method of integral transformations was applied, which made it possible to reduce the partial derivative problem to a Cauchy problem with a linear solution, which includes Bessel functions of the first kind and trigonometric functions. The final solution is given in the form of a rapidly converging alternating sign series (20).

Numerical simulation

A full numerical calculation of the temperature field inside the cylindrical structure was performed depending on the radial variable *r*, the variable *z*, and the time *t*.

For the numerical implementation of the analytical solution of the problem (20) of non-stationary thermal conductivity in a homogeneous cylinder with internal heat sources, the Python 3.11 programming language was used with the NumPy, SciPy, and Matplotlib libraries. The implementation includes the construction of a temperature field based on a series containing Bessel functions of the first kind and trigonometric functions, which is obtained by the method of finite integral transformations.

To increase the flexibility and interactivity of the analysis of the problem parameters, the ipywidgets module (Fig. 2) was used, which allows you to dynamically change the input physical characteristics (thermal conductivity, density, heat capacity, region sizes, source power, etc.) and instantly obtain graphical temperature dependences in characteristic cross-sections of the cylindrical region.

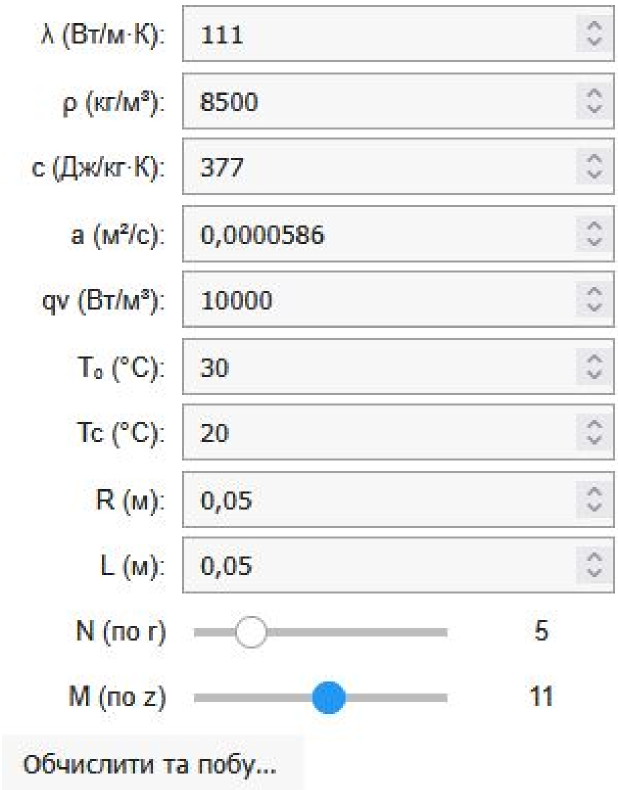


Fig. 2. The form of the ipywidgets module for dynamically changing input physical characteristics

The results were visualized using Matplotlib, with adaptive scaling of axes and color coding for comparing temperatures at different times. The eigenvalues were calculated using the jn_zeros() functions from the SciPy library, which implements the roots of Bessel functions.

Visualization of results

The graph in Fig. 3 shows how the temperature changes along the radius of the cylinder at different points in time (0, 75, 150, 225, 300 seconds). As a result of numerical simulation, intensive heating of the central part of the cylinder is observed, which is explained by the lack of heat exchange inside and the presence of constant internal heat sources. The temperature decreases smoothly towards the cylinder edges in both radial and axial directions, which is consistent with heat transfer through surfaces according to Newton's law. The edges of the cylinder (on the surface) lose less heat due to the small temperature gradient – the Newtonian effect at the boundaries is weak compared to the internal source. The graph shows a smooth change in temperature, without sharp jumps, which indicates a correct implementation of the model.

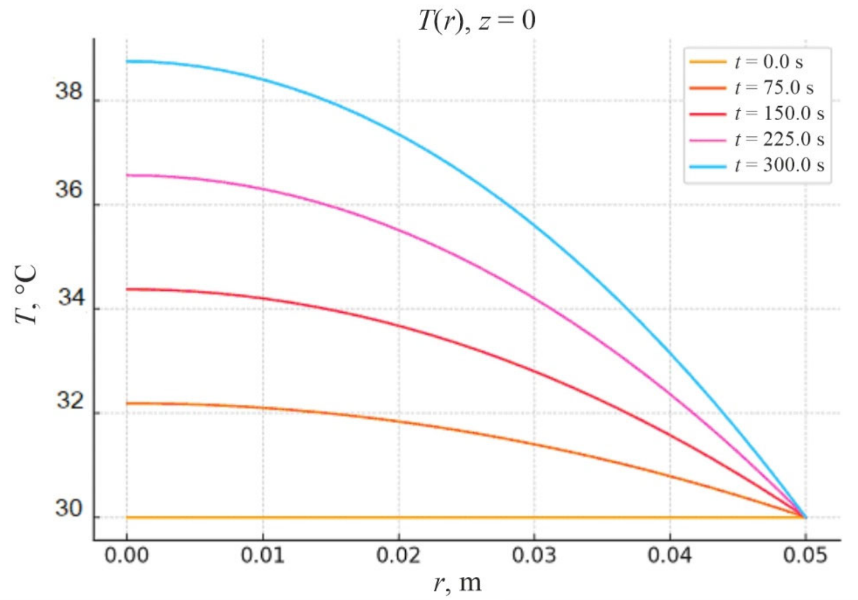


Fig. 3. Temperature distribution along the radius of the cylinder

Heat map of temperature *T*(*r*, *t*)

Temperature profiles were constructed for the dependences $T(r, z = 0, t)$, demonstrating the radial temperature distribution in the axial plane (Fig. 4).

The obtained graphs $T(r, t)$ (Fig. 4), 3D-profiles illustrate the nature of the thermal distribution and allow us to visualize the thermodynamic behavior of the system, demonstrating a uniform thermal evolution, which corresponds to the condition of material homogeneity and symmetry of the problem. The temperature heat map $T(r, t)$ simultaneously visualizes the temperature distribution along the radius and time, i.e. demonstrates the dynamics of heating the cylindrical structure along the radius R by internal heat sources. Bright areas – high temperatures, dark – cooler. It is clearly visible how the heating front advances: the temperature increases significantly first in the centre of the region, later – in more distant zones – the process of heat conduction from the core to the periphery occurs.

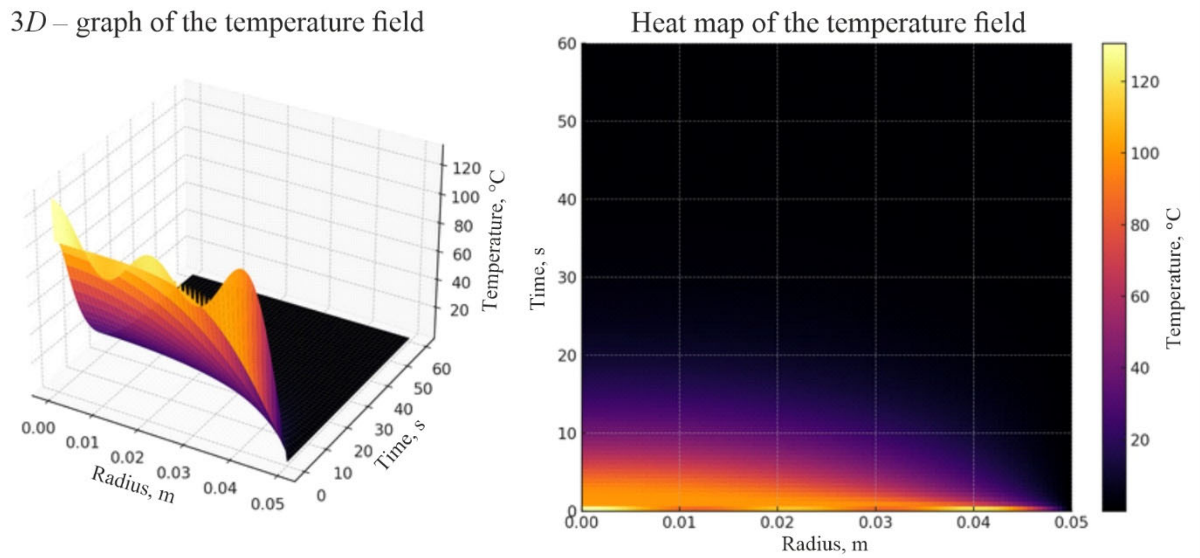


Fig. 4. 3D – graph and heat map of the temperature field

Numerical simulation and 3D visualization of the temperature field in a quarter-cylindrical region

The 3D graph displays the temperature distribution in the (r, z) plane at the initial point in time $t = 0$ s (Fig. 5 *a*) and at the maximum time $t = 150$ s (Fig. 5 *b*) – a central "core" with an increased temperature is visible, indicating the accumulation of heat in the central part of the body.

The obtained results confirm the physical essence of the problem: an internal heat source causes heating of the entire cylindrical region, with a maximum temperature in the central part. At the initial point in time, the temperature everywhere is equal to $T_0 = 20$ °C (Fig. 5 *a*). Due to the presence of internal heat sources and insulated side surfaces, the cylinder is heated from the inside.

The radial and axial temperature gradients are maintained throughout the heating process, demonstrating the efficiency of heat transfer at the boundaries.

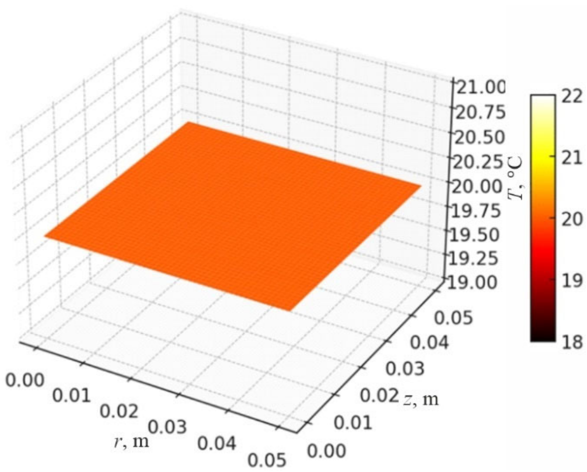
The model demonstrates rapid convergence, and within a short time the temperature increases noticeably.

The temperature behaviour matches the expected physical model: the area with the internal source heats up the most, especially near the centre.

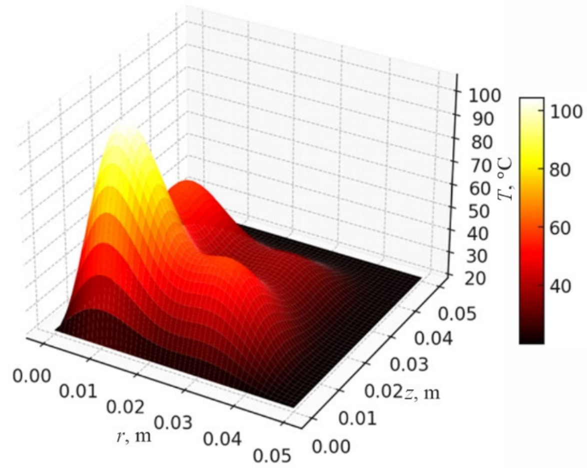
The gradual transfer of heat from the centre to the edges allows us to see the propagation of the thermal front in time. With increasing time, the amplitude of the temperature profile increases, and the curve becomes flatter – reflecting the decrease in the temperature gradient. The graphs (Fig. 5) confirm that the heat source in the central part of the cylinder creates a dominant heat wave, which gradually spreads to the edges.

The heat map $T(z, t)$ demonstrates a similar nature of heating along the axis of the cylinder. Due to symmetry and the same boundary conditions along z, the heating is uniform.

Numerical simulation confirmed the physical reliability and stability of the obtained analytical solution.



a)



b)

Fig. 5. Temperature field distribution (a – temperature distribution, t = 0 s; b – temperature distribution, t = 150 s)

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Conclusions

1. Using the method of finite integral transformations, an analytical solution to the problem of non-stationary thermal conductivity for a bounded cylinder with continuously operating heat sources was obtained.

2. A numerical calculation of the temperature field was performed, the behaviour of which corresponds to the proposed model, i.e. the central zone with an internal heat source heats up the most. Over time, the heating smoothly changes both in the radial and axial directions of the region.

3. The obtained promising solution provides accurate prediction of thermal processes in cylindrical TEGs with internal sources, which is the basis for engineering design of effective thermoelectric systems. The cylindrical shape provides the maximum temperature gradient between the centre (hot junction) and the periphery (cold junction), which is critically important for the efficient operation of the thermoelectric generator. The maximum temperature gradient

should be directed along the electrodes for effective use of the Seebeck effect. Internal temperatures should not exceed the limiting operating temperatures of the thermoelements.

4. The resulting visualizations provide a clear understanding of heat transfer processes and can be the basis for optimizing heat transfer in engineering systems.

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Аналітичне та чисельне дослідження температурних полів в циліндричних термоелементах з внутрішніми джерелами тепла

Методом скінченних інтегральних перетворень здійснено аналітичний розв'язок задачі нестаціонарної теплопровідності обмеженого циліндра з безперервно діючими джерелами тепла, вміщеного в середовище з постійною температурою. Запропоновано підхід числового дослідження теплових полів у циліндричних структурах з внутрішніми джерелами тепла з урахуванням впливу зовнішніх умов. Здійснено числову реалізацію аналітичного розв’язку, числове моделювання теплового розподілу та 3D-візуалізація термодинамічної поведінки системи. Методика числового моделювання використана для прогнозування температурного режиму в термоелектричних структурах з внутрішніми джерелами тепла, що є основою для інженерного проектування ефективних термоелектричних систем.

Ключові слова: нестаціонарна теплопровідність, метод кінцевих інтегральних перетворень, граничні умови, термоелектричний генератор, термоелектричні ефекти, внутрішні джерела тепла, температурне поле, числове моделювання.

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